


Reaction-diffusion systems

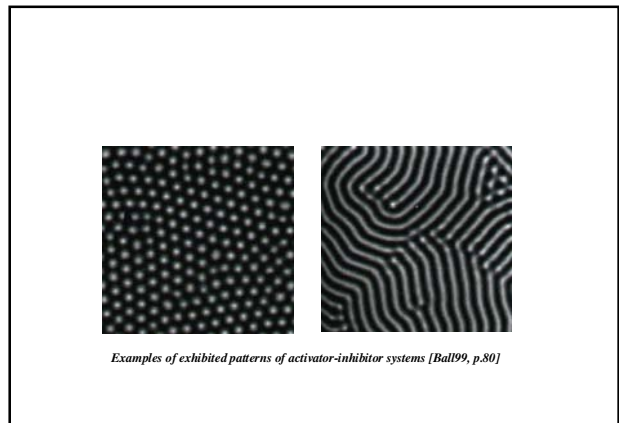
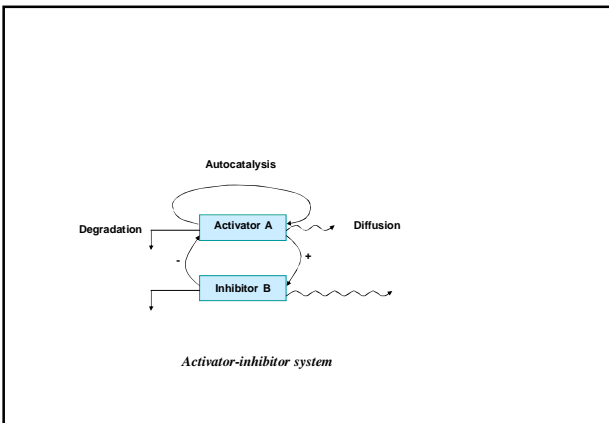
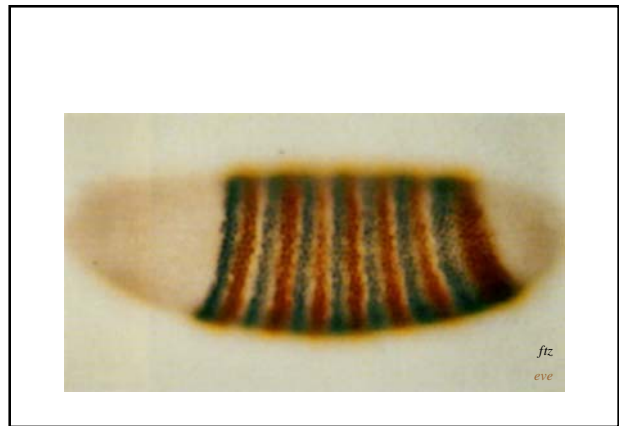
Turing and the consequences



Allan W. Turing

- "universal machine" (1935) (finite automata)
- "B-type unorganized machine" (1948) (neural network)
- "Chemical morphogenesis" (1952) (reaction-diffusion instability)

...be regarded by the machine as organized and by another as unorganized.
A typical example of an unorganized machine would be as follows.
The machine is made up from a rather large number N of similar units. Each unit has two input terminals, and has an output terminal which can be connected to the input terminals of other units. We may assume that for each integer r , $1 \leq r \leq N$ the outputs $i(r)$ and $o(r)$ are connected to the inputs



Reaction-diffusion model

$$\frac{\partial a}{\partial t} = f(a,b) + D_a \left(\frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} \right)$$

$$\frac{\partial b}{\partial t} = g(a,b) + D_b \left(\frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2} \right)$$

Activator-inhibitor patterns (Young's model)

Démo

The activator-substrate model

a_i — concentration of the activator
 s_i — concentration of the substrate

$$\frac{\partial a}{\partial t} = \rho a \left(\frac{a^2}{1+\kappa a^2} + \rho_0 \right) - \mu a + D_a \frac{\partial^2 a}{\partial x^2}$$

$$\frac{\partial s}{\partial t} = \sigma - \rho s \left(\frac{a^2}{1+\kappa a^2} + \rho_0 \right) - \nu s + D_s \frac{\partial^2 s}{\partial x^2}$$

Démo

Diffusion equation

first temporal derivative: rate

second spatial derivative: flux

$$\frac{\partial a}{\partial t} = D_a \nabla^2 a$$

a : concentration
 D_a : diffusion constant

Diffusion equation: temporal partial difference

a : concentration
 d : diffusion constant

$$\frac{\Delta a}{\Delta t} = \frac{a(t+1) - a(t)}{\Delta t} = d \left(\frac{a_{i-1} + a_{i+1}}{2} - a_i \right)$$

Diffusion equation: spatial partial difference

$a = f(x)$

$\Delta a / \Delta x$

$\Delta^2 a / \Delta x^2$

$$\frac{\Delta^2 a}{\Delta x^2} = a_{i-1} + a_{i+1} - 2a_i$$

Diffusion equation

first temporal derivative: rate

second spatial derivative: flux

$$\frac{\partial a}{\partial t} = D_a \nabla^2 a$$

a : concentration
 D_a : diffusion constant

The reaction-diffusion model

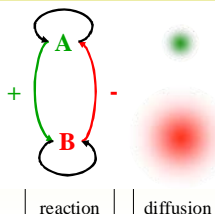
$$\frac{\partial a}{\partial t} = \underbrace{F(a,b)}_{\text{reaction}} + \underbrace{D_a \nabla^2 a}_{\text{diffusion}} \quad F(a,b) = a(b-1) - k_1$$

$$\frac{\partial b}{\partial t} = \underbrace{G(a,b)}_{\text{reaction}} + \underbrace{D_b \nabla^2 b}_{\text{diffusion}} \quad G(a,b) = k_2 - ab$$

Turing, A.M. (1952). The chemical basis of morphogenesis. Phil. Trans. Roy. Soc. London B 237: 37

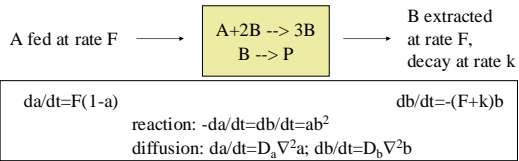
The reaction-diffusion model

$$\frac{\partial a}{\partial t} = \underbrace{F(a,b)}_{\text{reaction}} + \underbrace{D_a \nabla^2 a}_{\text{diffusion}}$$

$$\frac{\partial b}{\partial t} = \underbrace{G(a,b)}_{\text{reaction}} + \underbrace{D_b \nabla^2 b}_{\text{diffusion}}$$


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Reaction-diffusion: an example



$da/dt = F(1-a) - ab^2 + D_a \nabla^2 a$ a, b: concentrations of A, B

$db/dt = -(F+k)b + ab^2 + D_b \nabla^2 b$

Pearson, J. E.: Complex patterns in simple systems. Science 261, 189-192 (1993).

Discrete solution: finite difference equations

$$b'_{x,y} = b'_{x,y} + \Delta t \left\{ \begin{aligned} &D_b (b'_{x+1,y} + b'_{x-1,y} + b'_{x,y+1} + b'_{x,y-1} - 4b'_{x,y}) \\ &+ a'_{x,y} (b'_{x,y})^2 - (F+k)b'_{x,y} \end{aligned} \right\}$$

$$a'_{x,y} = a'_{x,y} + \Delta t \left\{ \begin{aligned} &D_a (a'_{x+1,y} + a'_{x-1,y} + a'_{x,y+1} + a'_{x,y-1} - 4a'_{x,y}) \\ &- a'_{x,y} (b'_{x,y})^2 + F(1-a'_{x,y}) \end{aligned} \right\}$$

Démo