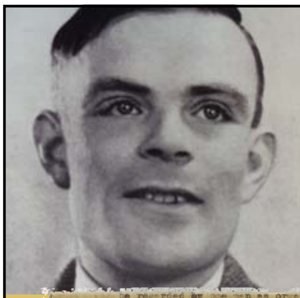
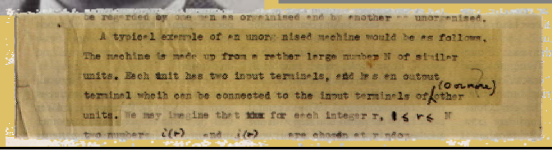
Reaction-diffusion systems

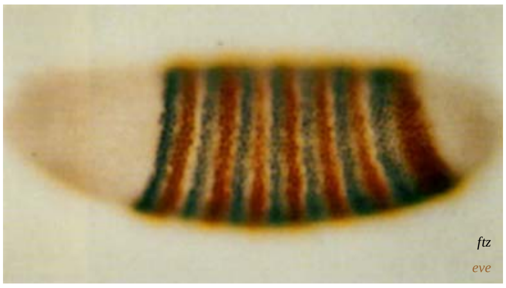
Turing and the consequences

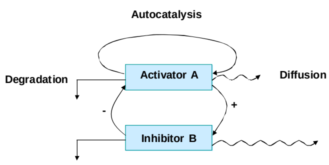


Allan W. Turing

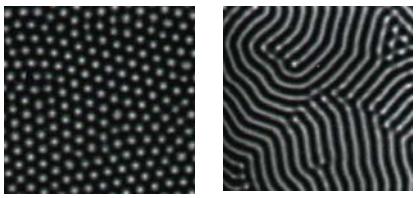
- "universal machine" (1935)
  (finite automata)
- "B-type unorganized machine" (1948)
  (neural network)
- "Chemical morphogenesis" (1952)
  (reaction-diffusion instability)



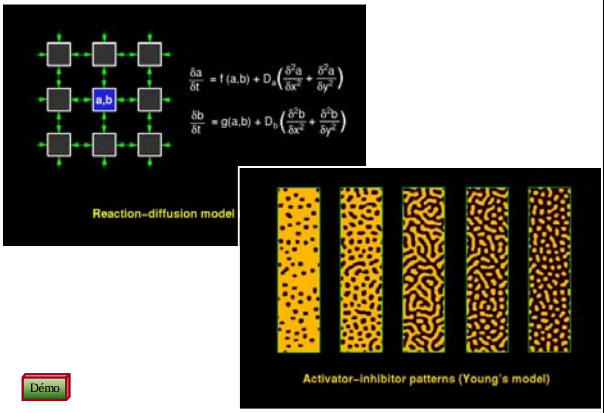




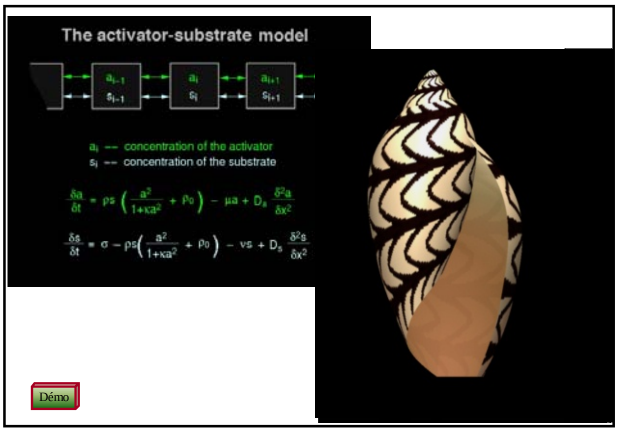
Activator-inhibitor system



Examples of exhibited patterns of activator-inhibitor systems [Ball99, p.80]



Démo



Démo

Diffusion equation

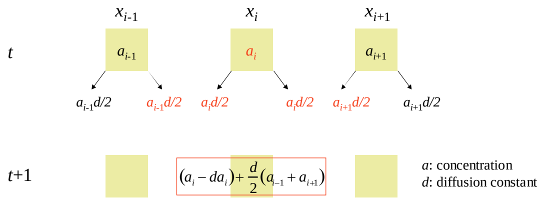
first temporal derivative: rate

second spatial derivative: flux

$$\frac{\partial a}{\partial t} = D_a \nabla^2 a$$

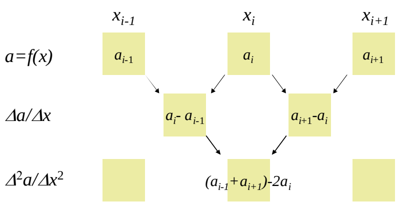
a: concentration
D_a: diffusion constant

Diffusion equation: temporal partial difference



$$\frac{\Delta a}{\Delta t} = \frac{a(t+1) - a(t)}{\Delta t} = d\left(\frac{a_{i-1} + a_{i+1}}{2} - a_i\right)$$

Diffusion equation: spatial partial difference



$$\frac{\Delta^2 a}{\Delta x^2} = a_{i-1} + a_{i+1} - 2a_i$$

Diffusion equation

first temporal derivative: rate

second spatial derivative: flux

$$\frac{\partial a}{\partial t} = D_a \nabla^2 a$$

a: concentration
D_a: diffusion constant

The reaction-diffusion model

$$\frac{\partial a}{\partial t} = F(a,b) + D_a \nabla^2 a \qquad F(a,b)=a(b-1)-k_1$$

$$\frac{\partial b}{\partial t} = G(a,b) + D_b \nabla^2 b \qquad G(a,b)=k_2-ab$$

reaction | diffusion

Turing, A.M. (1952). The chemical basis of morphogenesis. Phil. Trans. Roy. Soc. London *B* **237**: 37

The reaction-diffusion model

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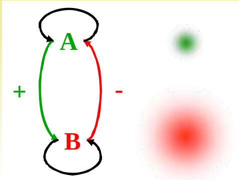

reaction | diffusion | reaction | diffusion

Turing, A.M. (1952). The chemical basis of morphogenesis. Phil. Trans. Roy. Soc. London *B* **237**: 37

Reaction-diffusion: an example

A fed at rate F ⟶ A+2B --> 3B; B --> P ⟶ B extracted at rate F, decay at rate k

da/dt=F(1-a) db/dt=-(F+k)b

reaction: -da/dt=db/dt=ab^2

diffusion: da/dt=$D_a\nabla^2 a$; db/dt=$D_b\nabla^2 b$

da/dt=F(1-a) - $ab^2 + D_a\nabla^2 a$

a,b: concentrations of A,B

db/dt=-(F+k)b + $ab^2 + D_b\nabla^2 b$

Pearson, J. E.: Complex patterns in simple systems. *Science* **261**, 189-192 (1993).

Discrete solution: finite difference equations

$$b_{x,y}^{t+1} = b_{x,y}^{t} + \Delta t \left\{ \begin{array}{l} D_b\left(b_{x+1,y}^t + b_{x-1,y}^t + b_{x,y+1}^t + b_{x,y-1}^t - 4b_{x,y}^t\right) \\ + a_{x,y}^t\left(b_{x,y}^t\right)^2 - (F+k)b_{x,y}^t \end{array} \right\}$$

$$a_{x,y}^{t+1} = a_{x,y}^{t} + \Delta t \left\{ \begin{array}{l} D_a\left(a_{x+1,y}^t + a_{x-1,y}^t + a_{x,y+1}^t + a_{x,y-1}^t - 4a_{x,y}^t\right) \\ - a_{x,y}^t\left(b_{x,y}^t\right)^2 + F\left(1 - a_{x,y}^t\right) \end{array} \right\}$$

Démo