Emergence in Complex Systems

Introduction to Evolutionary Game Theory

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Introduction

You already know that:

- Complex systems are systems made up of multiple agents
- The individual behaviors cannot be described.
- However, global behaviors **emerge** because of a propagation of the individual characteristics in evolution.

But you don't know any strict global mathematical explanation of this phenomenon! You don't even know if such a theory exists.

Such a framework exists: it is **Evolutionary Game Theory**.

Overview

1. **Introduction to Game Theory:** when two players are in a situation of simultaneous decision making, they can adopt optimal strategies (Nash equilibria).
2. **Evolutionary Stable Strategies:** when generalizing the classical games to populations, we notice that some equilibria are not stable when introducing different behaviors.
3. **Population dynamics:** the Evolutionary Stable Strategies are actually linked with mathematical properties of a very specific differential system (*replicator equation*)

Outline

Outline

What is Game Theory?

- A definition by Roger B. Myerson: *Game Theory can be defined as the study of mathematical models of conflict and cooperation between intelligent rational decision-makers.*
- The tools developed by game theory can be applied in any situation where two (or more) agents have to take decisions which have impacts over all other agents (**interdependent actions**).

Applications of Game Theory

Game Theory has applications in various domains:

- Economics
- Biology
- Artificial Intelligence
- Resource allocation and Network management
- Politics

Elk fights

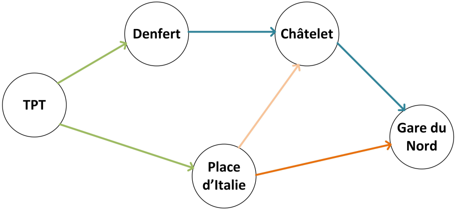
Biology

- Females are grouped in a harem held by a harem-holder male
- N other males want to access the harem
- A stranger entering the harem is chased away by the holder (fight)
- Males are wounded during the fight (depending on their strength)

Here, the individual strategy of each male depends on the *costs and gains* of a fight, but also on the *behavior of all other males*.

Congestion in public transports

Network management



- *N* students want to go from Télécom ParisTech to Gare du Nord
- The more people on the traject, the longer
- Three possible choices: (1) line 6 to Denfert then line B; (2) line 6 to Place d'Italie then line 5; (3) line 6 to Place d'Italie then line 7 to Châtelet then line B.

Pollution regulation

Politics / Economics

- N countries decide how much pollution they want to emit
- Each country n has a gain of $\beta_n(e_n)$ (benefits of emitting quantity e_n of pollution).
- Each country n has a deficit of $\phi_n(\sum_i e_i)$ because of the global pollution
- Which quantity will countries decide to emit?

Outline

Non-cooperative games and normal form

A game is called **non-cooperative** if the players make decisions independently. In a non-cooperative game, players cannot make coalitions or cooperation.

The **normal form** (also called **strategic form**) is the basic representation of a non-cooperative game.

Normal-form games

A normal-form consists in the description of:

- A set of players $I = \{1, \ldots, n\}$
- A family of strategy sets $(S^i)_{i \in I}$
- A family of payoff functions $g^i : \prod_{j \in I} S^j \mapsto \mathbb{R}$

The play is made up of two parts:

1. All players $i \in I$ choose **simultaneously** a strategy $s^i \in S^i$
2. Each player $i \in I$ gets a reward $g^i(s^1, \ldots, s^n)$

Remark: In the following, we will mainly consider *symmetric games* with two players, ie. games where the players have the same strategy sets: $S^1 = S^2 \triangleq S$

Examples

Coordination game

Two friends want to meet, either at place (A) or at place (B). They are satisfied only if they choose the same place

	A	B
A	$1,1$	$0,0$
B	$0,0$	$1,1$

Examples

Battle of the sexes

A married couple tries to decide what they will do this evening. The husband would rather watch football on TV (F) and the wife would rather go to the opera (O).

	F	O
F	$3,1$	$0,0$
O	$0,0$	$1,3$

Examples

Prisoner's dilemma

Two criminals are arrested by the police and are asked about their team. They have two choices: cooperate or defect.

	C	D
C	$-1, -1$	$-5, 0$
D	$0, -5$	$-3, -3$

Nash equilibrium in pure strategy

A Nash equilibrium in pure strategy is a couple of strategies $(p, q) \in S^2$ such that:

$$
\begin{aligned}
g^1(s, q) &\leq g^1(p, q) \qquad \forall s \in S \\
g^2(p, s) &\leq g^2(p, q) \qquad \forall s \in S
\end{aligned}
$$

In other words, no player benefits from a change of strategy.
Not all games have a Nash equilibrium in pure strategy.

Examples

Battle of the sexes

	F	O
F	$3,1$	$0,0$
O	$0,0$	$1,3$

Two Nash equilibria in pure strategy: (F,F) and (O,O).
None of these equilibria corresponds to a payoff maximum for both players.

Examples

Rock-Paper-Scissor

	R	P	S
R	$0,0$	$-1,1$	$1,-1$
P	$1,-1$	$0,0$	$-1,1$
S	$-1,1$	$1,-1$	$0,0$

There is no Nash equilibrium in pure strategy: for each strategy, at least one player may want to change its action.
What is the optimal way to play Rock-Paper-Scissor?

Mixed strategies and expected payoff

A **mixed strategy** is defined as a probability distribution over the pure strategies.

The set of mixed strategies is denoted $\Delta(S)$. For a finite number of strategies (of cardinal n), it corresponds to n-th dimensional simplex.

The expected payoff for player i corresponding to a couple of mixed strategies $(P, Q) \in \Delta(S)^2$ is defined as:

$$\mathbb{E}_{P,Q}[g^i(p,q)] = \sum_{p \in S} \sum_{q \in S} P(p)Q(q)g^i(p,q)$$

Mixed strategies and Nash equilibrium

Nash equilibrium in mixed strategy

A Nash equilibrium in mixed strategy is a couple of strategies $(P, Q) \in (\Delta(S))^2$ such that:

$$
\begin{aligned}
g^1(s, Q) &\leq g^1(P, Q) \qquad \forall s \in \Delta(S) \\
g^2(P, s) &\leq g^2(P, Q) \qquad \forall s \in \Delta(S)
\end{aligned}
$$

Nash theorem (1950)

For any finite game (finite number of players, finite number of strategies), there exists at least one Nash equilibrium in mixed strategy.

Example

Rock-Paper-Scissor

The strategy $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \in \Delta(S)$ defines a symmetric Nash equilibrium (ie. (p, p) is a Nash equilibrium) for the Rock-Paper-Scissor game.

The Hawk-Dove game

The players fight for a resource of value V. They can play two strategies:

- **Hawk:** trying to get the whole resource for himself
- **Dove:** not ready to fight for the resource

	H	D
H	$\frac{1}{2}(V-W), \frac{1}{2}(V-W)$	$V, 0$
D	$0, V$	$\frac{1}{2}V, \frac{1}{2}V$

The Hawk-Dove game

In the cas of $V < W$, there exists one single symmetric Nash equilibrium for the mixed strategy:

$$p = \left(\frac{V}{W}, 1 - \frac{V}{W} \right)$$

The game has also two pure Nash equilibria: (H, D) and (D, H) which are not biologically interesting (animals not labeled).

Outline

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Players in population games

Biological games involve large populations of species in the games.

At each generation, two randomly picked individuals can meet and play the game $\mathcal{G}$ together. If $\sigma \in \Delta(S)$ is a mixed strategy, the two following perspectives are equivalent:

- Each individual plays a mixed strategy σ
- Individuals play only pure strategies. For each pure strategy $i \in S$, a proportion σ_i of the population plays pure strategy i.

A mixed strategy can be interpreted as a portion of the population playing a pure strategy.

Introducing mutants in a population

Consider a population of N individuals playing a game $\mathcal{G}$ (eg. Hawk-Dove). Suppose that this population is programmed to play a strategy x.
We now introduce a mutant population (in proportion ϵ) playing with the strategy y.

Question:
What condition on x guarantees that the base population won't be affected by mutants?
This condition will define **Evolutionary Stable Strategies** (ESS).

Population dynamics

- At generation t: proportion ϵ_t of mutants playing strategy y
- Mean strategy: $q_t = (1 - \epsilon_t)x + \epsilon_t y$
- Total fitness of the majority strategy: $F + g(x, q_t)$
- Total fitness of the mutant strategy: $F + g(y, q_t)$
- Assume the following dynamics:

$$\epsilon_{t+1} = \frac{F + g(y, q_t)}{F + g(q_t, q_t)}\epsilon_t \quad \Leftrightarrow \quad \epsilon_t - \epsilon_{t+1} = \frac{g(q_t, q_t) - g(y, q_t)}{F + g(q_t, q_t)}\epsilon_t$$

Stability of the majority strategy
Intuition

Intuition: the majority strategy is stable if the mutant population vanishes:

$$\epsilon_t \underset{t \longrightarrow \infty}{\longrightarrow} 0$$

We want $\epsilon_t - \epsilon_{t+1} > 0$ ie $g(q_t, q_t) > g(y, q_t)$ which can be expanded (knowing the development of q_t in terms of x and y):

$$(1 - \epsilon_t)[g(x, x) - g(y, x)] + \epsilon_t[g(x, y) - g(y, y)] > 0$$

Stability of the majority strategy

Three cases

$$(1-\epsilon_t)[g(x,x)-g(y,x)]+\epsilon_t[g(x,y)-g(y,y)]>0$$

- If $g(x,x)<g(y,x)$: LHS cannot be negative for small values of ϵ
- If $g(x,x)>g(y,x)$: LHS can be negative for small enough values of ϵ. The mutant population vanishes.
- If $g(x,x)=g(y,x)$: the mutant population vanishes if and only if $g(x,y)>g(y,y)$.

Outline

Definition of an ESS

Definition

An **Evolutionary Stable Strategy** of a symmetric two-person game $\mathcal{G} = \langle S, g \rangle$ is a strategy $x \in \Delta(S)$ satisfying the following conditions:

1. *Equilibrium condition:* (x, x) is a Nash equilibrium.
2. *Stability condition:* Every best reply y to x different from x satisfies $g(x, y) > g(y, y)$

Remarks:

- An ESS doesn't always exist for symmetric two-players games.

Examples

Prisoners dilemma

The unique Nash Equilibrium (D,D) is also an ESS.

Harm thy neighbor

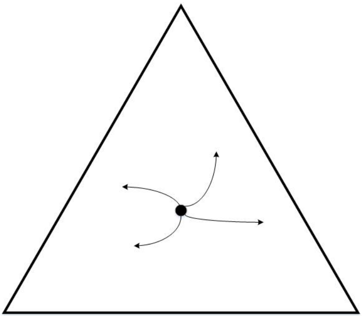
	A	B
A	$2,2$	$1,2$
B	$2,1$	$2,2$

(A,A) and (B,B) are both Nash equilibria, but only (B,B) is an ESS.

Examples

Rock-Paper-Scissor

The only Nash equilibrium $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is not an ESS.
You can prove it with the definition of the ESS, but would you be able to explain it?



Outline

Outline

Description of the replicator model

- Consider a population of N individuals (called replicators)
- Each individual plays only one pure strategy
- Each individual passes its strategy to its descendants
- The number of descendants depends linearly of the mean gain of the parent (called fitness)
- The standard birth (resp. death) ratio is β (resp. δ)
- The number of individuals playing i-th strategy is denoted by $n_i = x_i N$

Population evolution

- Replicators playing i-th strategy:

$$\dot{n}_i = (\beta - \delta + g(s_i, \mathbf{x}))\, n_i$$

- Global population:

$$\dot{N} = \sum_{i=1}^{n} \dot{n}_i = (\beta - \delta) \underbrace{\sum_{i=1}^{n} n_i}_{=N} + \underbrace{\left(\sum_{i=1}^{n} x_i g(s_i, \mathbf{x}) \right)}_{=g(\mathbf{x},\mathbf{x})} N$$

- Link between individuals and global population:

$$\dot{n}_i = N\dot{x}_i + x_i \dot{N}$$

Replicator equation

From the previous results, we get:

$$N\dot{x}_i = (\beta - \delta + g(s_i, \mathbf{x}))x_i N - x_i(\beta - \delta + g(\mathbf{x}, \mathbf{x}))N$$

And finally:

Replicator equations

$$\boxed{\dot{x}_i = (g(s_i, \mathbf{x}) - g(\mathbf{x}, \mathbf{x}))\, x_i}$$

Outline

Fixed points in replicator dynamics

Definition: Fixed point

A **fixed point** of the replicator dynamics is a strategy that satisfies $\dot{x}_i = 0$ for all pure strategy i.

Definition: Asymptotically stable fixed point

A fixed point $\mathbf{x}^*$ of the replicator dynamics is called **asymptotically stable** if for each pure strategy $i \in S$ there exists $\delta > 0$ such that $|x_i(0) - x_i^*| < \delta$ implies $\lim_{t \longrightarrow \infty} |x_i(t) - x_i^*| = 0$.

Fixed points and Nash equilibrium

Theorem
If $\sigma \in \Delta(S)$ is a mixed strategy of $\mathcal{G}$ such that (σ, σ) is a symmetric Nash equilibrium, then the state $\mathbf{x} = \sigma$ is a fixed point of the replicator equation.

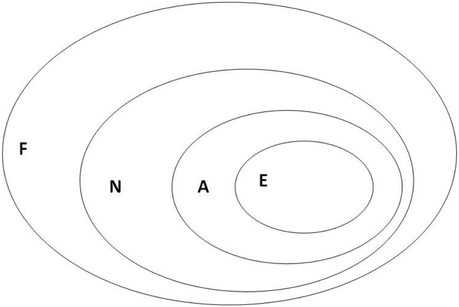
Theorem
If $\mathbf{x}$ is an asymptotically stable fixed point of the replicator equation and $\sigma \in \Delta(S)$ is the mixed strategy of $\mathcal{G}$ associated to the state $\mathbf{x}$, then the symmetric strategy (σ, σ) is a symmetric Nash equilibrium.

Theorem
If $\sigma \in \Delta(S)$ is a mixed strategy of the game $\mathcal{G}$ such that (σ, σ) is an ESS of $\mathcal{G}$, then the state $\mathbf{x}$ associated to σ is an asymptotically stable fixed point of the corresponding replicator equation.

Summary of the theorems

If $\mathcal{G}$ is a game, we consider the following sets:

- **F**: fixed points of the replicator equation
- **N**: symmetric Nash equilibrium of $\mathcal{G}$
- **A**: asymptotically stable points of the replicator equation
- **E**: ESS of $\mathcal{G}$



Outline

Replicator for two-strategies games

Suppose that the played game $\mathcal{G}$ has only two strategies. We can choose the notations $x_1 = x$ and $x_2 = 1 - x$.
The replicator equation can be rewritten as:

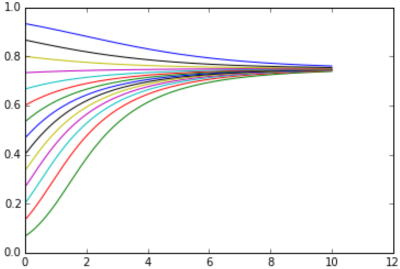
$$\begin{aligned} \dot{x} &= (g(s_1, x) - g(\mathbf{x}, \mathbf{x}))\, x \\ &= (g(s_1, \mathbf{x}) - x g(s_1, \mathbf{x}) - (1 - x) g(s_2, \mathbf{x}))\, x \\ &= x(1 - x)[g(s_1, \mathbf{x}) - g(s_2, \mathbf{x})] \end{aligned}$$

Hawk-Dove

Replicator equation:

$$\dot{x} = \frac{1}{2}x(1-x)(V-Wx)$$

Solutions of the replicator equation for $V = 3$ and $W = 4$:

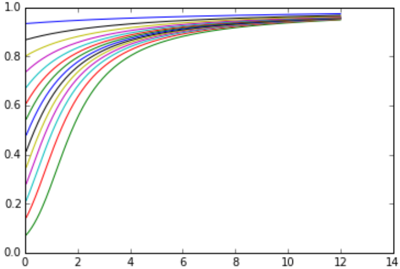


Hawk-Dove

Replicator equation:

$$\dot{x} = \frac{1}{2}x(1-x)(V-Wx)$$

Solutions of the replicator equation for $V = 4$ and $W = 4$:



Chicken game

Chicken game

	Fearless	*Safe*
Fearless	$-\alpha, -\alpha$	$\gamma, -\delta$
Safe	$-\delta, \gamma$	$-\beta, -\beta$

Replicator equation:

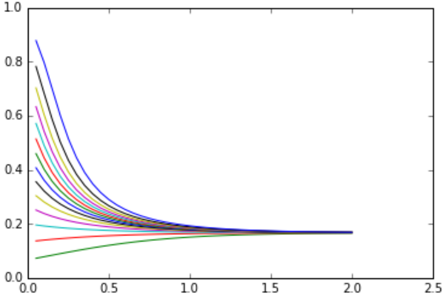
$$\dot{x} = x(1-x)[(\beta-\alpha-\gamma-\delta)x+\gamma-\beta]$$

Chicken

Replicator equation:

$$\dot{x} = x(1-x)[(\beta-\alpha-\gamma-\delta)x+\gamma-\beta]$$

Solutions of the replicator equation for $\alpha = 10, \beta = 2, \gamma = 5$ and $\delta = 5$:



Prisoner's dilemma

	C	D
C	α, α	$0, \beta$
D	$\beta, 0$	γ, γ

Replicator equation:

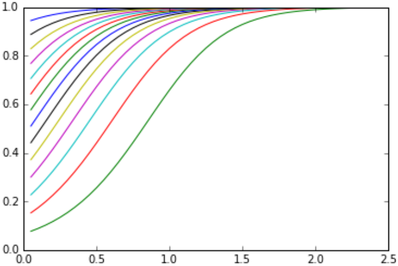
$$\dot{x} = x(1-x)[(\alpha - \beta + \gamma)x - \gamma]$$

Prisoner's dilemma

Replicator equation:

$$\dot{x} = x(1-x)[(\alpha - \beta + \gamma)x - \gamma]$$

Solutions of the replicator equation for $\alpha = -1, \beta = -5$ and $\gamma = -3$:

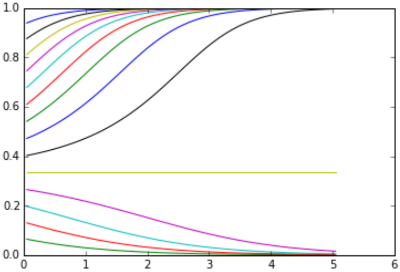


Prisoner's dilemma

Replicator equation:

$$\dot{x} = x(1 - x)[(\alpha - \beta + \gamma)x - \gamma]$$

Solutions of the replicator equation for $\alpha = 5, \beta = 3$ and $\gamma = 1$:



Rock-Paper-Scissor

	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

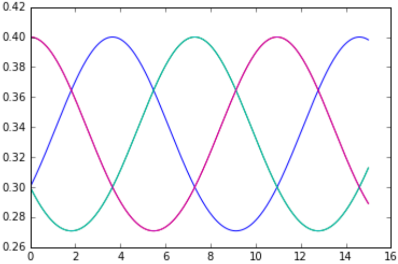
$$g(s_1, \mathbf{x}) = x_3 - x_2$$
$$g(s_1, \mathbf{x}) = x_1 - x_3$$
$$g(s_1, \mathbf{x}) = x_2 - x_1$$
$$g(\mathbf{x}, \mathbf{x}) = x_1(x_3 - x_2) + x_2(x_1 - x_3) + x_3(x_2 - x_1)$$

Rock-Paper-Scissor

Replicator equation:

$$\dot{x}_i = [g(s_i, \mathbf{x}) - g(\mathbf{x}, \mathbf{x})]x_i$$

Solutions of the replicator equation with initial state $x = \frac{1}{10}(3, 3, 4)$**:**

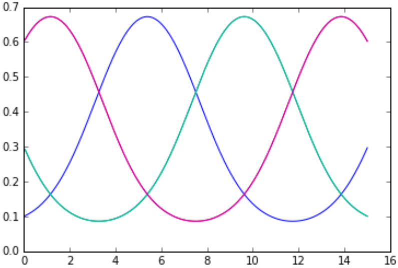


Rock-Paper-Scissor

Replicator equation:

$$\dot{x}_i = [g(s_i, \mathbf{x}) - g(\mathbf{x}, \mathbf{x})]x_i$$

Solutions of the replicator equation with initial state $x = \frac{1}{10}(1, 3, 6)$:



Rock-Paper-Scissor

Variant 1: Unequal scores

	R	P	S
R	$0, 0$	$-\beta, \alpha$	$\alpha, -\beta$
P	$\alpha, -\beta$	$0, 0$	$-\beta, \alpha$
S	$-\beta, \alpha$	$\alpha, -\beta$	$0, 0$

$$g(s_1, \mathbf{x}) = \alpha x_3 - \beta x_2$$

$$g(s_1, \mathbf{x}) = \alpha x_1 - \beta x_3$$

$$g(s_1, \mathbf{x}) = \alpha x_2 - \beta x_1$$

$$g(\mathbf{x}, \mathbf{x}) = x_1(\alpha x_3 - \beta x_2) + x_2(\alpha x_1 - \beta x_3) + x_3(\alpha x_2 - \beta x_1)$$

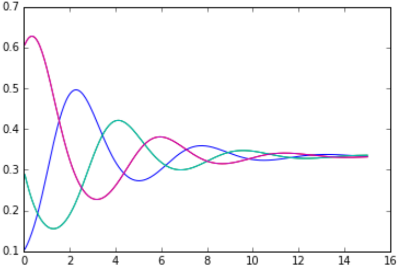
Rock-Paper-Scissor

Variant 1: Unequal scores

Replicator equation:

$$\dot{x}_i = [g(s_i, \mathbf{x}) - g(\mathbf{x}, \mathbf{x})]x_i$$

Solutions of the replicator equation with initial state $x = \frac{1}{10}(1, 3, 6)$ and parameters $\alpha = 3, \beta = 1$:



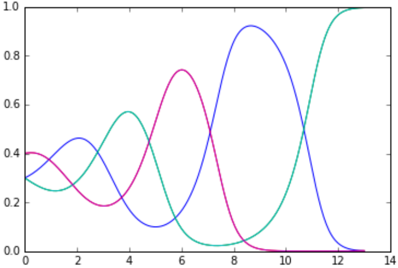
Rock-Paper-Scissor

Variant 1: Unequal scores

Replicator equation:

$$\dot{x}_i = [g(s_i, \mathbf{x}) - g(\mathbf{x}, \mathbf{x})]x_i$$

Solutions of the replicator equation with initial state $x = \frac{1}{10}(3, 3, 4)$ and parameters $\alpha = 1, \beta = 3$:



Rock-Paper-Scissor

Variant 2: Rock-Paper-Scissor-Lizard-Spock

Rock-Paper-Scissor

Variant 2: Rock-Paper-Scissor-Lizard-Spock

	R	*P*	*Sc*	*L*	*Sp*
R	$0, 0$	$-\beta, \alpha$	$\alpha, -\beta$	$\alpha, -\beta$	$-\beta, \alpha$
P	$\alpha, -\beta$	$0, 0$	$-\beta, \alpha$	$-\beta, \alpha$	$\alpha, -\beta$
Sc	$-\beta, \alpha$	$\alpha, -\beta$	$0, 0$	$\alpha, -\beta$	$-\beta, \alpha$
L	$-\beta, \alpha$	$\alpha, -\beta$	$-\beta, \alpha$	$0, 0$	$\alpha, -\beta$
Sp	$\alpha, -\beta$	$-\beta, \alpha$	$\alpha, -\beta$	$-\beta, \alpha$	$0, 0$

Rock-Paper-Scissor

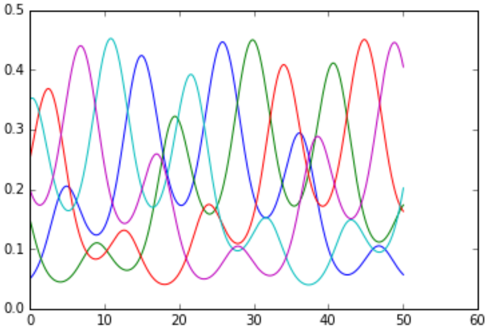
Variant 2: Rock-Paper-Scissor-Lizard-Spock

$$g(s_1, \mathbf{x}) = \alpha(x_3 + x_4) - \beta(x_2 + x_5)$$
$$g(s_2, \mathbf{x}) = \alpha(x_1 + x_5) - \beta(x_3 + x_4)$$
$$g(s_3, \mathbf{x}) = \alpha(x_2 + x_4) - \beta(x_1 + x_5)$$
$$g(s_4, \mathbf{x}) = \alpha(x_2 + x_5) - \beta(x_1 + x_3)$$
$$g(s_5, \mathbf{x}) = \alpha(x_1 + x_3) - \beta(x_2 + x_4)$$
$$g(\mathbf{x}, \mathbf{x}) = x_1 g(s_1, \mathbf{x}) + x_2 g(s_2, \mathbf{x}) + x_3 g(s_3, \mathbf{x}) + x_4 g(s_4, \mathbf{x}) + x_5 g(s_5, \mathbf{x})$$

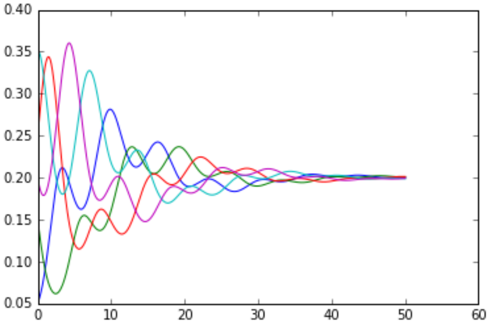
Rock-Paper-Scissor

Variant 2: Rock-Paper-Scissor-Lizard-Spock

Solutions of the replicator equation with initial state $x = \frac{1}{100}(5, 15, 25, 35, 20)$ and parameters $\alpha = 1, \beta = 1$:



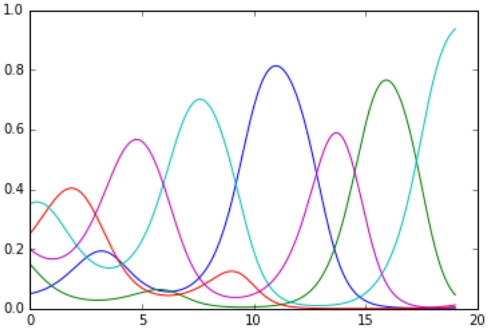
Solutions of the replicator equation with initial state $x = \frac{1}{100}(5, 15, 25, 35, 20)$ and parameters $\alpha = 2, \beta = 1$:



Rock-Paper-Scissor

Variant 2: Rock-Paper-Scissor-Lizard-Spock

Solutions of the replicator equation with initial state $x = \frac{1}{100}(5, 15, 25, 35, 20)$ and parameters $\alpha = 1, \beta = 2$:



Outline

Conclusion

- Evolutionary population effects can be studied with the perspective of game theory
- Traditional game theory introduces the notion of *Nash equilibrium* which describe compromise in social interactions
- Evolutionary game theory describes the convergence to Nash equilibria
- Emergence phenomenons in complex systems are not magical effects: they are the result of a convergence to a Nash equilibrium
- Equilibrium points exist in all systems, but they cannot necessarily emerge

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